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ANTIPLANE DEFORMATION OF AN ELASTIC WEDGE UNDER ACTION CONCENTRATED NEAR THE CORNER POINT†

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The asymptotic behaviour of the remote field in a wedge under static and dynamic loading in the neighbourhood of the vertex is studied. It is shown that, unlike the well-known Carothers' paradox, neither the solution of the static nor the dynamic problem satisfy St Venant's principle for any wedge angle. The solution of the dynamic problem for a harmonic action is constructed using Sommerfeld's integrals. It is shown that the asymptotic behaviour of the solution in a remote zone is not of St Venant type, i.e. the corresponding coefficient of the leading term of the asymptotic expansion is equal to a "fractional" moment of the external forces.

The solution of the static plane problem of the theory of elasticity for a wedge with normal stresses applied to its sides in a small neighbourhood of the corner point, the stresses being statically equivalent to a pair of forces, is remarkable in that for a fairly large wedge the asymptotic behaviour of the stresses in a remote region is inconsistent with the solution of St Venant's limiting problem (Carothers' paradox [1]). It is interesting to see whether or not the paradox is carried over to the case of dynamic (impulse or harmonic) loading of a wedge. Below we answer this question in the case of antiplane shear. The question of the specific nature of St Venant's principle as applied to dynamic problems has been considered in [2–8].

1. It is useful to precede the solution of the dynamic problem of the theory of elasticity for a wedge by a completely elementary analysis of the corresponding static problem. In the case of antiplane shear the displacement vector has only one non-zero component $u_3 = w(r, \theta)$, which satisfies the Laplace equation in $\Omega = \{(r, \theta\, z)\,|\, r > 0, -\alpha < \theta < \alpha, -\infty < z < \infty\}$. We prescribe the stresses

$$\theta = \pm\alpha, \quad \tau_{\theta z} = \mu r^{-1} dw/d\theta = f(r) \tag{1.1}$$

on the faces of the wedge, assuming that the support of $f(r)$ is concentrated in a small neighbourhood $(0, \delta)$ of the wedge vertex, the principal force vector being equal to zero and the principal moment being non-zero

$$F = M_0 = \int_0^\delta f(r)dr = 0, \quad M_1 = \int_0^\delta rf(r)dr \neq 0 \tag{1.2}$$

To solve the above problem we use the Mellin integral transform

$$\tilde{w}(s,\theta) = \int_0^\infty w(r,\theta) r^{s-1} dr$$

The Mellin image $\tilde{w}(s, \theta)$ satisfies the following equation and boundary condition

$$\partial^2 \tilde{w} / \partial\theta^2 + s^2 \tilde{w} = 0 \tag{1.3}$$

$$\theta = \pm\alpha, \quad \mu\partial\tilde{w} / \partial\theta = \int_0^\infty f(r) r^2 dr \equiv M_s$$

We will confine ourselves to considering the antisymmetric solution. Then

$$\tilde{w} = M_s (\mu s)^{-1} \sin(s\theta) / \cos(s\alpha)$$

and so the inverse Mellin transform gives the displacement

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$$w(r,\theta)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{M_s\sin(s\theta)}{\mu s\cos(s\alpha)}r^{-s}ds \tag{1.4}$$

We obtain an asymptotic estimate of the integral (1.4) for $r \gg 1$ by completing the integration contour in (1.4) by a semicircle of infinitely large radius in the half-plane Re $s > 0$ and using the residue theorem. The leading term of the asymptotic expression has the form

$$w(r,\theta)=-\frac{M_\alpha\sin(\pi\theta/(2\alpha))}{(\mu\pi/(2\alpha))r^{\pi/(2\alpha)}}+o(r^{-\pi/(2\alpha)}) \tag{1.5}$$

$$M_\alpha=\int_0^\infty f(r)r^{\pi/(2\alpha)}dr$$

The coefficient M_α cannot be interpreted as the moment of forces, and so St Venant's principle is not satisfied in its traditional form in the case in question, even though the integral dependence on the boundary interaction (as a moment with fractional power) is preserved. The solution is of St Venant type for $\alpha = \pi/2$, i.e. for an elastic half-plane. Note that the solution of the limiting problem concerned with antiplane shear of a wedge by a pair of forces concentrated at the vertex exists only in this case and cannot be constructed for $\alpha \neq \pi/2$.

2. For harmonic action

$$\theta=\pm\alpha,\quad \tau_{\theta z}=f(r)e^{-iwt} \tag{2.1}$$

on the wedge faces the problem can be reduced to the Helmholtz equation

$$\Delta w+k^2w=0,\quad k=\omega\sqrt{\rho/\mu} \tag{2.2}$$

We shall seek an antisymmetric solution of the equation satisfying the radiation condition and a condition on the edge of the wedge [9] in the form of Sommerfeld's integral

$$w(r,\theta)=\frac{1}{2\pi i}\int_C[\Phi(\varsigma+\theta)-\Phi(\varsigma-0)]e^{ikr\cos\varsigma}d\varsigma \tag{2.3}$$

where the contour C contains the half-strip $0 < \text{Re}\,\zeta < \pi$, Im $\zeta > 0$. It can be shown by direct verification that the integral (2.3) satisfies the Helmholtz equation (2.2). As regards the boundary condition (2.1), using the inversion formulae [10]

$$F(\varsigma)=ik\sin\varsigma\int_0^\infty f(r)e^{-ikr\cos\varsigma}dr,\quad f(r)=\frac{1}{2\pi i}\int_C F(\varsigma)e^{ikr\cos\varsigma}d\varsigma \tag{2.4}$$

we obtain the following functional equation for $\Phi(\zeta)$

$$\Phi(\varsigma+\theta)+\Phi(\varsigma-\theta)=\sigma(\varsigma),\quad \sigma(\varsigma)=\frac{F(\varsigma)}{i\mu k\sin\varsigma}=\frac{1}{\mu}\int_0^\infty f(r)e^{-ikr\cos\varsigma}dr \tag{2.5}$$

i.e. Malyuzhinets' equation, which can be solved using the Fourier integral transform. After transformations, we have

$$\Phi(\varsigma)=\frac{i}{4\alpha}\int_{-i\infty}^{i\infty}\sigma(\eta)\left[\cos\frac{\pi}{2\alpha}(\varsigma-\eta)\right]^{-1}d\eta \tag{2.6}$$

It follows that the solution of (2.1), (2.2) can be represented in the form

$$w(r,\theta)=\int_C\int_{-i\infty}^{i\infty}\sigma(\eta)A(\varsigma,\theta,\eta)e^{ikr\cos\varsigma}d\eta d\varsigma \tag{2.7}$$

$$8\pi\alpha A(\varsigma,\theta,\eta)=\left[\cos\frac{\pi}{2\alpha}(\varsigma+\theta-\eta)\right]^{-1}-\left[\cos\frac{\pi}{2\alpha}(\varsigma-\theta-\eta)\right]^{-1}$$

Changing the order of integration in (2.7), we find that

$$w(r,\theta)=\frac{1}{\mu}\int_0^\infty f(\rho)P(r,\rho,\theta)d\rho$$

$$P(r,\rho,\theta)=\int_C \int_{-i\infty}^{i\infty} A(\varsigma,\theta,\eta)e^{ik(-\rho\cos\eta+r\cos\varsigma)}d\eta d\varsigma$$

We recall that $f(\rho)$ is assumed to be non-zero over the interval $(0, \delta)$ and relationships (1.2) are satisfied. Therefore, assuming δ to be small compared with the wavelength $\lambda = 2\pi/k$ and retaining the leading term in ρ in the asymptotic expansion of $P(r, \rho, \theta)$, we obtain

$$w=\frac{1}{\mu}\int_0^\delta f(\rho)\rho^{\pi/(2\alpha)}d\rho Q(r,\theta) \tag{2.8}$$

with some bounded function $Q(r, \theta)$. The asymptotic form (2.8) of the solution of the dynamic problem is not of St Venant type.†

In the case of non-stationary action

$$\theta=\pm\alpha, \quad \tau_{\theta z}=f(r)q(t)$$

one can use the Fourier integral representation

$$q(t)=\operatorname{Re}\int_0^\infty Q(k)e^{-ikct}dk\left(c=\sqrt{\frac{\mu}{\rho}}\right)$$

of $q(t)$, where c is the velocity of transverse waves in the elastic medium. We will assume that the spectral action density $Q(k)$ differs significantly from zero in the interval $(0, k_0)$ with $k_0\delta \ll 2\pi$. Then in the remote region $kr \gg 1$ the asymptotic form of the solution of the non-stationary problem has the form

$$w(r,\theta,t)=\operatorname{Re}\int_0^\infty Q(k)w(r,\theta,k)e^{-ikct}dk$$

where $w(r, \theta\ k)$ depends on the fractional moment of the external force M_α.

The problems for the Laplace and Helmholtz equations in a wedge considered above also have other physical interpretations in hydrodynamics, electrodynamics, heat conduction theory, etc.

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†Formula (2.7) in [11] is analysed incorrectly.